Analysis 1A — Supremum Example

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# Question

Here is an example of finding the supremum of a set taken from an old problem sheet, together with three possible methods to find it.

Example 1

Let

Show that is bounded above and find

## Method 1 — Contradiction

Solution.

First, note for :

Hence is bounded above by . Therefore, by the completeness axiom, as exists and

Next, suppose for contradiction that . Now, for any

Taking and applying Archimedes' Postulate, such that

which is a contradiction as Hence , and by combining our found inequalities, .

## Method 2 — Alternative Characterisation of Suprema

Solution.

First, note for :

Hence is bounded above by . Therefore as the completeness axiom says that exists.

We claim that Fix Then, for

Now, by Archimedes' Postulate, such that , from which

At this stage, take . Since was arbitrary, we have that such that So, by the alternative characterisation of suprema (Theorem 2.1),

## Method 3 — Limits

Note that this doesn’t work in general, but it might be quicker when you can use it. It relies on the following theorem (which we’ll eventually cover):

Theorem 1

A bounded, increasing sequence is convergent, and its limit is given by

Solution.

Define for .

**Step 1 — Show is bounded above**:

First, note for :

Hence is bounded above by . Therefore, as the completeness axiom says that exists.

**Step 2 — Show is increasing (i.e. show )**:

We have for ,

So is increasing. Hence, by the above theorem, converges, and by the *Algebra of Limits*,

as expected!